Energy bursts in fiber bundle models of composite materials

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A bundle of many fibers with stochastically distributed breaking thresholds for the individual fibers is considered as a model of composite materials. The bundle is loaded until complete failure, to capture the failure scenario of composite materials under external load. The fibers are assumed to share the load equally, and to obey Hookean elasticity right up to the breaking point. We determine the distribution of bursts in which an amount of energy E is released. The energy distribution follows asymptotically a universal power law $E^{-5/2}$, for any statistical distribution of fiber strengths. A similar power law dependence is found in some experimental acoustic emission studies of loaded composite materials.

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I. INTRODUCTION

During the failure process of composite materials under external load, avalanches of different magnitudes are produced, where an avalanche consists of simultaneous rupture of several elements. Such avalanches cause a sudden internal stress redistribution in the material, and are accompanied by a rapid release of mechanical energy. A useful experimental technique to monitor the energy release is to measure the acoustic emissions, the elastically radiated waves produced in the bursts [1–5].

Fiber bundles with statistically distributed thresholds for breakdown of individual fibers are interesting models of failure processes in materials. They are characterized by simple geometry and clear-cut rules for how stress caused by a failed element is redistributed on the intact fibers. The interest of these models lies in the possibility of obtaining exact results, thereby providing inspiration and reference systems for studies of more complicated materials. (For reviews, see [6–10].) The statistical distribution of the *size* of avalanches in fiber bundles is well studied [11–14], but the distribution of the burst *energies* is not. In this paper we therefore determine the statistics of the energies released in fiber bundle avalanches.

We study equal-load-sharing models, in which the load previously carried by a failed fiber is shared equally by all the remaining intact fibers in the bundle [15–18]. We consider a bundle consisting of a large number N of elastic fibers, clamped at both ends (see Fig. 1). The fibers obey Hooke's law, such that the energy stored in a single fiber at elongation x equals $1/2x^2$, where we for simplicity have set the elasticity constant equal to unity. Each fiber i is associated with a breakdown threshold x_i for its elongation. When the length exceeds x_i the fiber breaks immediately, and does not contribute to the strength of the bundle thereafter. The individual thresholds x_i are assumed to be independent random variables with the same cumulative distribution function $P(x)$ and a corresponding density function $p(x)$:

$$\mathrm{Prob}(x_i < x) = P(x) = \int_0^x p(y)dy. \tag{1}$$

At an elongation x the total force on the bundle is x times the number of intact fibers. The average, or macroscopic, force is given by the expectation value of this,

$$\langle F\rangle = Nx[1 - P(x)]. \tag{2}$$

In the generic case $\langle F\rangle$ will have a single maximum F_c, a critical load corresponding to the maximum load the bundle can sustain before complete breakdown of the whole system. The maximum occurs at a critical value x_c for which $d\langle F\rangle/dx$ vanishes. Thus x_c satisfies

$$1 - P(x_c) - x_c p(x_c) = 0. \tag{3}$$

II. ENERGY STATISTICS

Let us characterize a burst by the number n of fibers that fail, and by the lowest threshold value x among the n failed fibers. The threshold value $x_{\max}$ of the strongest fiber in the burst can be estimated to be

$$x_{\max} \simeq x + \frac{n}{Np(x)}, \tag{4}$$

since the expected number of fibers with thresholds in an interval Δx is given by the threshold distribution function as $Np(x)\Delta x$. The last term in (4) is of the order $1/N$, so for a

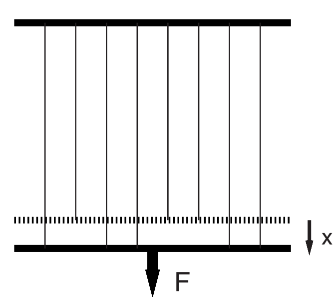

FIG. 1. The fiber bundle model.

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very large bundle the differences in threshold values among the failed fibers in one burst are negligible. Hence the energy released in a burst of size n that starts with a fiber with threshold x is given with sufficient accuracy as

$$E = \tfrac{1}{2}nx^2. \tag{5}$$

In a statistical analysis of the burst process, Hemmer and Hansen [11] calculated the expected number of bursts of size n, starting at a fiber with a threshold value in the interval $(x,x+dx)$, as

$$f(n,x)dx = N\frac{n^{n-1}}{n!}\frac{1-P(x)-xp(x)}{x}X(x)^n e^{-nX(x)}dx, \tag{6}$$

with the abbreviation

$$X(x) = \frac{xp(x)}{1-P(x)}. \tag{7}$$

The expected number of bursts with energies less than E is therefore

$$G(E) = \sum_n \int_0^{\sqrt{2E/n}} f(n,x)dx, \tag{8}$$

with a corresponding energy density

$$g(E) = \frac{dG}{dE} = \sum_n (2En)^{-1/2} f(n,\sqrt{2E/n}). \tag{9}$$

Explicitly,

$$g(E) = N\sum_n g_n(E), \tag{10}$$

with

$$g_n(E) = \frac{n^{n-1}}{2En!}[1-P(s)-sp(s)]\left[\frac{sp(s)}{1-P(s)}\exp\left(-\frac{sp(s)}{1-P(s)}\right)\right]^n. \tag{11}$$

Here

$$s \equiv \sqrt{2E/n}. \tag{12}$$

With a critical threshold value x_c, it follows from (5) that a burst energy E can be obtained only if n is sufficiently large,

$$n \geq 2E/x_c^2. \tag{13}$$

Thus the sum over n starts with

$$n = 1 + [2E/x_c^2]; \tag{14}$$

here $[a]$ denotes the integer part of a.

We discuss now both the high-energy and the low-energy behavior of the energy density $g(E)$.

A. High-energy asymptotics

Bursts with high energies correspond to bursts in which many fibers rupture. In this range we may use Stirling's approximation for the factorial $n!$, replace $1+[2E/x_c^2]$ by $2E/x_c^2$, and replace the summation over n by an integration. Thus

$$g(E) \simeq \frac{N}{2E^{3/2}\pi^{1/2}}\int_{2E/x_c^2}^{\infty}\frac{e^n}{n^{3/2}}[1-P(s)-sp(s)] \times\left[\frac{sp(s)}{1-P(s)}\exp\left(-\frac{sp(s)}{1-P(s)}\right)\right]^n dn, \tag{15}$$

where s is the abbreviation (12). By changing integration variable from n to s we obtain

$$g(E) \simeq \frac{N}{2E^{3/2}\pi^{1/2}}\int_0^{x_c}[1-P(s)-sp(s)] \times\left[\frac{sp(s)}{1-P(s)}\exp\left(1-\frac{sp(s)}{1-P(s)}\right)\right]^n ds = \frac{N}{2E^{3/2}\pi^{1/2}}\int_0^{x_c}[1-P(s)-sp(s)]e^{-Eh(s)}ds, \tag{16}$$

with

$$h(s) \equiv \left(-\frac{1-P(s)-sp(s)}{1-P(s)} + \ln\frac{1-P(s)}{sp(s)}\right)\frac{2}{s^2}. \tag{17}$$

For large E the integral (16) is dominated by the integration range near the minimum of $h(s)$. At the upper limit $s=x_c$ we have $h(x_c)=0$, since $1-P(x_c)=x_c p(x_c)$, Eq. (3). This is also a minimum of $h(s)$. To see that, note that with $y\equiv 1-sp(s)/[1-P(s)]$, the expression in large parentheses in (17) is of the form

$$-y-\ln(1-y) = y^2 + O(y^3), \tag{18}$$

with a minimum at $y=0$.

In a systematic expansion about the maximum of the integrand in (16), at $s=x_c$, the first factor in the integral (16) vanishes linearly,

$$1-P(s)-sp(s) = (x_c-s)[2p(x_c)+x_c p'(x_c)] + O(x_c-s)^2, \tag{19}$$

and, as we have seen, $h(s)$ has a quadratic minimum,

$$h(s) \simeq \left(\frac{2p(x_c)+x_c p'(x_c)}{x_c^2 p(x_c)}\right)^2 (x_c-s)^2. \tag{20}$$

Inserting these expressions into (16) and integrating, we obtain the following asymptotic expression:

$$g(E) \simeq N\frac{C}{E^{5/2}}, \tag{21}$$

where

$$C = \frac{x_c^4 p(x_c)^2}{4\pi^{1/2}[2p(x_c)+x_c p'(x_c)]}. \tag{22}$$

In Figs. 2 and 3 we compare the theoretical formula with simulations for the uniform distribution,

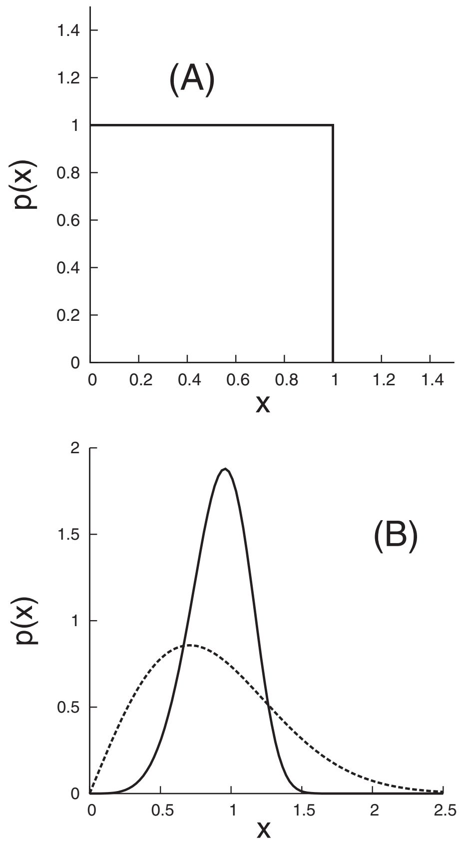

FIG. 2. (a) Uniform threshold distribution (23) and (b) Weibull distribution (24) of index 2 (dotted line) and 5 (solid line).

$$P(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1, \\ 0 & \text{for } x > 1, \end{cases} \tag{23}$$

which corresponds to $x_c = \frac{1}{2}$ and $C = 2^{-7}\pi^{-1/2}$, and for the Weibull distribution with index $k=2$,

$$P(x) = 1 - e^{-x^k} \quad \text{for } x \geq 0, \tag{24}$$

which corresponds to $x_c = 2^{-1/2}$ and $C = 2^{-5}(2\pi e)^{-1/2}$.

The corresponding asymptotics (21) are also exhibited in Fig. 3. For both threshold distributions the agreement between the theoretical asymptotics and the simulation results is very satisfactory. The exponent $-5/2$ in the energy burst distribution is clearly universal. Note that the asymptotic distribution of the burst magnitudes n is governed by the same exponent [11].

B. Low-energy behavior

The low-energy behavior of the burst distribution is by no means universal: $g(E)$ may diverge, vanish, or stay constant as $E \to 0$, depending on the nature of the threshold distribution. In Fig. 4 we exhibit simulation results for the low-energy part of $g(E)$ for the uniform distribution and the Weibull distributions of index 2 and index 5.

We see that $g(E)$ approaches a finite limit in the Weibull $k=2$ case, approaches zero for Weibull $k=5$, and apparently diverges in the uniform case. All this is easily understood, since bursts with low energy predominantly correspond to single fiber bursts ($n=1$, i.e., $E = x^2/2$) and to fibers with low threshold values. The number of bursts with energy less than E therefore corresponds to the number of bursts with $x < \sqrt{2E}$, which is close to $NP(\sqrt{2E})$. This gives

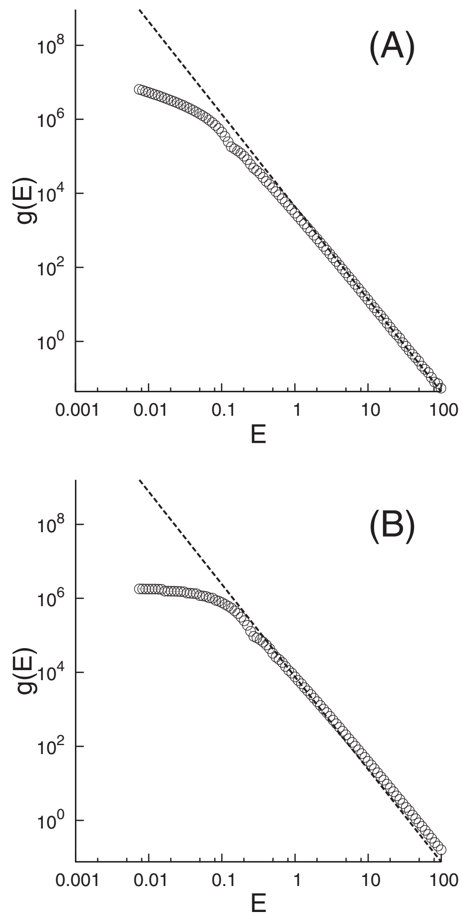

FIG. 3. Simulation results for $g(E)$ characterizing energy bursts in fiber bundles with (a) the uniform threshold distribution (23) and (b) the Weibull distribution (24) of index 2. The graphs are based on 1000 samples with $N = 10^6$ fibers in each bundle. Open circles represent simulation data, and dashed lines are the theoretical results (21) and (22) for the asymptotics.

$$g(E) \simeq N \frac{p(\sqrt{2E})}{\sqrt{2E}} \quad \text{when } E \to 0. \tag{25}$$

For the uniform distribution $g(E)$ should therefore diverge as $(2E)^{-1/2}$ for $E \to 0$. The simulation results in Fig. 4 are consistent with this divergence. For the Weibull distribution of index 2, on the other hand, (25) gives $g(E) \to 2N$ when $E \to 0$, a value in agreement with simulation results in the figure. Note that for a Weibull distribution of index k, the low-energy behavior is $g(E) \propto E^{(k-2)/2}$. Thus the Weibull distribution with $k=2$ is a borderline case between divergence and vanishing of the low-energy density.

The same lowest-order results can be obtained from the general expression (10), which also can provide more detailed low-energy expansions.

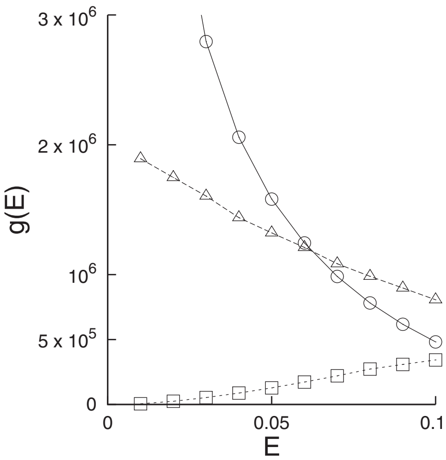

FIG. 4. Simulation results for the burst distribution $g(E)$, in the low-energy regime, for the uniform threshold distribution (circles), the Weibull distribution with $k=2$ (triangles), and the Weibull distribution with $k=5$ (squares). The graphs are based on 1000 samples with $N=10^6$ fibers in each bundle.

III. SUMMARY

In the present paper we have studied the distribution of burst energies during the failure process in fiber bundles with statistically distributed thresholds for breakdown of individual fibers. We have derived an exact expression for the energy density distribution $g(E)$, and shown that for high energies the energy density obeys a power law with exponent $-5/2$. This asymptotic behavior is universal, independent of the threshold distribution. A similar power law dependence is found in some experimental observations on acoustic emission studies [1,2] of loaded composite materials.

In contrast, the low-energy behavior of $g(E)$ depends crucially on the distribution of the breakdown thresholds in the bundle. $g(E)$ may diverge, vanish, or stay constant for $E\rightarrow 0$.

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